

PL-TR-92-2181

AD-A257 772



2

DIFFERENTIAL CARTOGRAPHY

J.D. Zund

New Mexico State University
Department of Mathematical Sciences
Las Cruces, NM 88003-0001

July 13, 1992

DTIC
ELECTE
OCT 20 1992
S A D

Scientific Report No. 4

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED



PHILLIPS LABORATORY
Directorate of Geophysics
AIR FORCE MATERIEL COMMAND
HANSCOM AIR FORCE BASE, MA 01731-5000

92 1


3

92-27446



This technical report has been reviewed and is approved for publication.


CHRISTOPHER JEKELI
Contract Manager


THOMAS P. ROONEY
Branch Chief


DONALD H. ECKHARDT
Division Director

This document has been reviewed by the ESD Public Affairs Office (PA) and is releasable to the National Technical Information Service (NTIS).

Qualified requestors may obtain additional copies from the Defense Technical Information Center. All others should apply to the National Technical Information Service.

If your address has changed, or if you wish to be removed from the mailing list, or if the addressee is no longer employed by your organization, please notify IMA, Hanscom AFB, MA 01731-5000. This will assist us in maintaining a current mailing list.

Do not return copies of this report unless contractual obligations or notices on a specific document requires that it be returned.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
<small>Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.</small>				
1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE 13 July 1992		3. REPORT TYPE AND DATES COVERED Scientific Report No. 4
4. TITLE AND SUBTITLE Differential Cartography			5. FUNDING NUMBERS PE 61102F PR 2309 TA G1 WU BZ Contract F19628-89-K-0044	
6. AUTHOR(S) J. D. Zund				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) New Mexico State University Department of Mathematical Sciences Las Cruces, NM 88003-0001			8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) Phillips Laboratory Hanscom AFB, MA 01731-5000 Contract Manager: Christopher Jekeli/GPEG			10. SPONSORING / MONITORING AGENCY REPORT NUMBER PL-TR-92-2181	
11. SUPPLEMENTARY NOTES				
12a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; Distribution unlimited			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) The various properties of conformal projections of a sphere and a surface of revolution are derived by using the notions of differential cartography. It is shown that familiar properties of the Mercator projection may be carried over from a sphere to an arbitrary surface of revolution.				
14. SUBJECT TERMS Differential cartography Mercator projection			15. NUMBER OF PAGES 16	
			16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT SAR	

CONTENTS

1.	Introduction	1
2.	The Mercator Projection of a Sphere	3
3.	The Mercator Projection for Surfaces of Revolution	6
4.	Discussion	7
	References	9

Accession For	
NTIS CRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and/or Special
A-1	

Preface

The following report is a written version of an abstract "Differential Cartography" which was presented on August 21, 1991 at the XX General Assembly of the International Association of Geodesy in Vienna, Austria, in a working session of the Special Commission: Geodetic Aspects of the Law of the Sea under the chairmanship of Professor Petr Vaniček (Canada).

The abstract addresses the various properties of conformal projections of a sphere and seeks to generalize them to an arbitrary smooth surface of revolution by using the basic notions of a new discipline which we call differential cartography. In particular, it is shown how the familiar properties of the classical Mercator projection may be carried over from a sphere, or spheroid, to an arbitrary surface of revolution.

"DIFFERENTIAL CARTOGRAPHY"

Joseph Zund

Summary:

The various properties of conformal projections of a sphere and a surface of revolution are derived using the notions of differential cartography. It is shown that familiar properties of the Mercator projection may be carried over from a sphere to an arbitrary surface of revolution.

1. Introduction

Differential cartography may be regarded as a reformulation of mathematical cartography from the viewpoint of differential geometry. Mathematical cartography may be roughly reduced to two types of problems: the

Direct Problem: Construct a projection, hence a map, having certain specified properties; and the

Indirect Problem: Given certain properties, determine all projections having these properties.

In practice, the Direct Problem has primarily been the one which has attracted the most attention. Many fascinating projections are known and their number seems limitless, and restricted only by the ingenuity of their discoverers. However, such projections need not immediately reveal the underlying structural features which govern their construction. The Indirect Problem is concerned with remedying this situation. Few general results are known, and differential cartography is intended to provide a general approach for handling this problem. Indeed, our primary goal in the present paper is to understand the Mercator projection, its principal properties, and the degree of generality intrinsic in its construction. In particular, we will show that this conformal projection can be extended from spheres to arbitrary surfaces of revolution which include spheroids as a special case.

This projection was employed in 1569 by Gerardus Mercator in his world atlas: *Nova et accurata orbis terrae descriptio ad usum navigantium emendate accommodata*. His world map measured 2×1.32 meters, extended from latitude 80° N to $66^\circ 3'S$, exhibited parallels and meridians intersecting at right angles, and a special latitude law of scale especially devised to yield the loxodrome property. A *loxodrome*, or *rhumb line*, on a sphere is a curve which intersects the meridians at a constant angle, and corresponds to the mariner's task of sailing courses of constant heading. Mercator's problem was to find a projection having the *loxodrome property* that such courses project into straight lines on a planar map.

Mercator's methodology is unknown, but it was undoubtedly graphical since in his

time none of the mathematics (e.g. logarithms and the differential/integral calculus) was available. The *angle-preserving* character, or *conformality*, of his projection was only gradually recognized, however it is now recognized as a fundamental property of this projection. For example, each loxodrome intersects every meridian at a constant angle, and under a conformal projection angles are preserved; it necessarily follows that the images of the parallels and meridians are orthogonal, and the image of a loxodrome intersects the meridians at the given fixed angle. Hence, one may immediately conclude that the images of many loxodromes at such fixed angles are parallel straight lines.

Our approach employing differential cartography is essentially a modern rendition of classical theory. Lambert (1772) first suggested the construction of maps based on the use of the angle-preserving property. Euler (1777) then gave the detailed mathematical theory of the Mercator projection, and then proved the impossibility of obtaining an isometric, i.e. distance-preserving, representation of the sphere on the Euclidean plane. Lagrange (1779) considered the general problem of conformal projections for surfaces of revolution. The term 'conformal' was introduced by Gauss (1822, 1843) who studied the mapping problem for a pair of arbitrary surfaces, indicated the geodetic applications, and the rich connection with the theory of functions of a complex variable. The results in Section 2 are implicit in the work of Lagrange and Gauss. They were well known in the last century, but are not commonly employed in contemporary accounts. Section 3 considers the extension of these results for surfaces of revolution. I have not found them explicitly stated in the classical literature, however they are in the spirit of Lagrange and Gauss and would have been no surprise to them. Our exposition concludes with a discussion in Section 4.

The terminology employed in our work essentially follows that employed in classical differential geometry, e.g. in the textbook of Struik (1961), however even in the mathematical literature the usage of some terms is not uniform. Unfortunately, often the mathematical usage of many terms is not consistent with that utilized in cartography.

2. The Mercator Projection of a Sphere

The following discussion is given with some detail since it illustrates the basic approach which will be presented in Section 3.

Let S_2 denote a unit sphere in ordinary Euclidean 3-space E_3 . We choose the Gaussian parametrization of S_2 , in terms of the longitude ω and latitude ϕ . Then the *line element*, or the *first fundamental form*, of S_2 is

$$ds^2 = \cos^2 \phi d\omega^2 + d\phi^2 \quad (1)$$

and (ω, ϕ) determines an *orthogonal parameter net* on S_2 . This net can be put in a non-trivial *isometric*, or *isothermic*, form (see page 171 of Struik (1961)) by rewriting (1) in the form

$$ds^2 = \cos^2 \phi \{d\omega^2 + \sec^2 \phi d\phi^2\} \quad (2)$$

which amounts to choosing a unit scale along the parallels or ω -curves, and a variable scale on the meridians or ϕ -curves on S_2 .

It is now convenient to replace (ω, ϕ) by a pair of *real isotropic parameters* (ξ, η) . These are obtained by employing a complex factorization of (2):

$$ds^2 = \cos^2 \phi (d\omega + i \sec \phi d\phi)(d\omega - i \sec \phi d\phi)$$

and introducing the *complex variable*

$$\zeta := \xi + i\eta. \quad (3)$$

Then

$$d\zeta = d\xi + i d\eta = d\omega + i \sec \phi d\phi$$

and so upon omitting the constants of integration we have

$$\xi = \omega \quad (4)$$

$$\eta = \int \sec \phi d\phi = \log \tan (\phi/2 + \pi/4).$$

Thus the *isotropic isothermic form* of ds^2 becomes

$$ds^2 = \operatorname{sech}^2 \eta (d\xi^2 + d\eta^2) \quad (5)$$

since $\cos \phi = \operatorname{sech} \eta$.

We now consider a *conformal projection*, or *transformation*, T of S_2 onto the

Euclidean plane $\hat{\hat{E}}_2$, where 'hats' are used on quantities to emphasize that they occur as T-images. Upon choosing Cartesian coordinates (\hat{x}, \hat{y}) in $\hat{\hat{E}}_2$, the line element is accordingly

$$d\hat{s}^2 = d\hat{x}^2 + d\hat{y}^2. \quad (6)$$

It is well-known from complex function theory that conformal transformations are defined by *analytic functions*, indeed the conformality of a transformation may be taken as defining the analyticity of the function describing it. Let F denote an analytic function of ζ , then it is easy to check that

$$d\hat{s}^2 = |dF|^2 = \cosh^2 \eta |F_\zeta|^2 ds^2 \quad (7)$$

where the subscript on F indicates differentiation. The analyticity of F is guaranteed by the *Cauchy-Riemann equations*, which become $F_{\bar{\zeta}} = 0$ where the bar denotes complex conjugation. Since in differential geometry a conformal transformation between a pair of surfaces S and \hat{S} is defined by their metric tensors being proportional, the *same* Gaussian parametrization must be taken on both surfaces. In our case this requires identifying (ξ, η) and (\hat{x}, \hat{y}) so (4) yields

$$\begin{aligned} \hat{x} &:= \xi = \omega \\ \hat{y} &:= \eta = \log \tan (\phi/2 + \pi/4). \end{aligned} \quad (8)$$

The *linear distortion* ϵ_1 of $T : S_2 \rightarrow \hat{\hat{E}}_2$ is defined by $\epsilon_1 := d\hat{s}/ds$, and so by (7)

$$\epsilon_1 = \cosh \eta |F_\zeta| = \sec \phi |F_\zeta|. \quad (9)$$

The most familiar kinds of conformal projections T in mathematical cartography then correspond to the following choices of the analytic function F :

- a) $F = \zeta$, $T = T_M$ (*Mercator*);
- b) $F = ce^{i\zeta}$, where c is a constant, $T = T_\Sigma$ (*stereographic*);
- c) $F = ce^{ik\zeta}$ where c is a constant and $k > 0$ is a real number, $T = T_L$ (*Lambert*).

Note that for T_M , F is merely the identity function -- a deceptively simple result based on a rather complicated ζ as in (4) -- and $\epsilon_1 = \sec \phi$.

The basic results about T_M may now be stated as follows:

- (i) $T_M : S_2 \rightarrow \hat{E}_2$ has the loxodrome property;
- (ii) The only $T : S_2 \rightarrow \hat{E}_2$ having the loxodrome property is $T = T_M$.

The proof of (i) is immediate. A loxodrome \mathcal{C} on S_2 intersecting the meridians at an angle ϑ in the (ξ, η) parameters satisfies $d\xi/d\eta = \tan \vartheta$, and so is given by an isotropic equation

$$\mathcal{C} : \xi = \eta \tan \vartheta + c_1 ;$$

which is equivalent to

$$\omega \cot \vartheta = \log \tan (\phi/2 + \pi/4) + c_2 ;$$

and whose T-image is

$$\hat{\mathcal{C}} : \hat{y} = \hat{x} \cot \vartheta + c_3$$

where c_1, c_2, c_3 are constants. The equation of $\hat{\mathcal{C}}$ is a straight line in \hat{E}_2 . Note that although $\hat{\mathcal{C}}$ encircles a pole of S_2 in spiral-like fashion an infinite number of times, it has a finite length. For example if P and Q are the respective initial and final points on \mathcal{C} , then the length of \mathcal{C} is

$$L(\mathcal{C}) = (\phi_Q - \phi_P) \sec \vartheta .$$

Hence if P lies on the equator ($\phi_P = 0$), and Q is the (North) pole ($\phi_Q = \pi/2$) and

$$L(\mathcal{C}) = \frac{\pi}{2} \sec \vartheta .$$

Finally, we note that for T_Σ , the T-image of \mathcal{C} is an equi-angular spiral in \hat{E}_2 .

The proof of (ii) is more subtle and depends on the properties of isothermic and isotropic parameter systems. These include the fact that all pairs of real isothermic parameters are related by analytic functions, and that isotropic parameters provide a particularly convenient parametrization of S_2 . Indeed, a necessary and sufficient condition that a transformation T between a pair of surfaces be conformal is that their isotropic curves correspond. This is the content of our equation (8).

3. The Mercator Projection for Surfaces of Revolution

We now extend the results of Section 2 for an arbitrary surface of revolution S_R . We suppose that S_R is generated by the revolution of the graph of a smooth function f of a radial variable about an axis ℓ in E_3 . Choosing ℓ to be the z -axis, this radius variable becomes $\sqrt{x^2 + y^2}$ and is denoted by ρ . Then the longitude ω and ρ may be taken as Gaussian parameters of S_R , and in terms of the parameter net (ω, ρ) the line element of S_R becomes

$$ds^2 = \rho^2 d\omega^2 + (1 + f'^2) d\rho^2 \quad (11)$$

where the dash denotes differentiation with respect to ρ . An isothermic form of ds^2 is given by

$$ds^2 = \rho^2 \left\{ d\omega^2 + \frac{(1+f'^2)}{\rho^2} d\rho^2 \right\} \quad (12)$$

and the corresponding isotropic parameters (ξ, η) are

$$\begin{aligned} \xi &:= \omega \\ \eta &:= \int \frac{1}{\rho} \sqrt{1+f'^2} d\rho. \end{aligned} \quad (13)$$

As in (3) we may introduce the complex variable ζ , and taking g to be the inverse function, $\rho := g(\eta)$, of the second equation in (13) we have the isotropic isothermic form of ds^2 :

$$\begin{aligned} ds^2 &= \rho^2 (d\xi^2 + d\eta^2) \\ &= g^2 (d\xi^2 + d\eta^2) \\ &= g^2 |d\zeta|^2. \end{aligned} \quad (14)$$

Let $T : S_R \rightarrow \hat{E}_2$ denote a conformal transformation, so as in Section 2 we have

$$d\hat{s}^2 = \frac{1}{g^2} |F_\zeta|^2 ds^2 \quad (15)$$

where

$$\begin{aligned} \hat{x} &:= \xi \\ \hat{y} &:= \eta. \end{aligned} \quad (16)$$

The linear distortion of T is then

$$\varepsilon_1 = \frac{1}{g} |F_\zeta|. \quad (17)$$

Note that in (17) the function g characterizes the particular form of S_R , and F is an arbitrary analytic function of ζ .

The analysis in Section 2, including the specializations of F , indicated in (10) carry over to define T_M , T_Σ , and T_L respectively. Hence, we immediately have the basic results:

- (i) $T_M : S_R \rightarrow \hat{E}_2$ has the loxodrome¹ property,
- (ii) The only $T : S_R \rightarrow \hat{E}_2$ having the loxodrome property is $T = T_M$.

The proofs of (i) and (ii) are completely analogous to the reasoning given in Section 2 when (4) is replaced by (13). It would be truly remarkable that the above results were not established by the mathematical cartographers of the nineteenth century. However, we have been unable to locate specific references in the literature which establish this. In any case, our approach which is based on differential cartography and properties of analytic functions may well be new.

4. Discussion

The results in Section 3 may be surprising, however, and if this is so, it is because the familiar results on the Mercator projection in Section 2 are not usually discussed as an Indirect Problem. The viewpoint of differential cartography, indeed *la raison d'être* for it, is to provide a geometric approach to mathematical cartography. In this respect, the key ingredients leading to the results (i) and (ii) are the emphasis on the isothermal form and isotropic parameters for the surface line element. These notions are purely of a geometric character and their full power, e.g. their connection with complex function theory, would seem to be evident only within the framework offered by differential cartography.

¹The notion of a loxodrome can be generalized for an S_R to be a curve which intersects a pencil of planes at a constant angle. The axis ℓ of S_R is then taken as the axis of the pencil which consists of meridional planes.

Three conclusions are immediately suggested by the analysis given in Sections 2 and 3. First, by viewing the conformal projection as being defined by an analytic function F of a complex variable, it is clear that the three cases indicated in (10) represent only the *simplest* choices of F . Indeed, on function-theoretic grounds it could be argued that these choices are almost the simplest possible analytic functions. Hence, the Mercator, stereographic, and Lambert projections by no means exhaust the possibilities. Thus, the use of conformal projections in mathematical cartography is hardly a fully explored technique, until other choices of F are investigated.

The second conclusion is that the results for S_R offer the possibility of replacing the spheroid by a pear-shaped globe having a hump at the North pole and a depression at the South pole *provided* that these irregularities can be regarded as being axially symmetric. In any case, one could devise a conformal projection which could be more accurate than the usual spheroidal model. Indeed, one could employ T_M in such a case by taking the base parallel not as the equator ($\phi = 0$), but as a higher latitude $\phi = \phi^0$. Recently Vaniček and Sjöberg (1991) have considered an analogous generalization of zonal harmonics in seeking a new description of the gravity field of the Earth.

Finally, (ii) of Sections 2 and 3 show that for conformal transformations T_M is the *only* projection having the loxodrome property. However, this property requires that angle ϑ be preserved under a mapping of S_2 , or S_R , into \hat{E}_2 , hence it is meaningful only for a conformal transformation. Thus, we may conclude that the loxodrome property can hold *only* for a conformal transformation T , and as already noted by our item (ii) we must have $T = T_M$. In other words, the loxodrome property *uniquely* characterizes Mercator transformations.

I am indebted to Professor Petr Vaniček for his suggestion to investigate this question, and to Bernard Chovitz for his careful scrutiny of a preliminary version of the manuscript and several useful discussions.

References

- Euler, L. (1777). De representatione superficei sphaericae super plano, Acta acad. scient. Petropolitanae 107-132 = *Leonhardi Euleri Opera Omnia, Commentationes Geometricae* III, editit A. Speiser, Orell Füssli, Turici (1955) 248-275.
- Gauss, C.F. (1822). Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer gegebenen Flächen so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird, Astron. Abh. herausgegeben v. H. C. Schumacher, Heft III = *Carl Friedrich Gauss Werke*, Bd 4, Königl. Ges. der Wiss. zu Göttingen (1880), 189-216.
- Gauss, C.F. (1843). Untersuchungen über Gegenstände der Höheren Geodäsie, Erste Abhandlung, Abh. der Königl. Ges. der Wiss. zu Göttingen II = *Carl Friedrich Gauss Werke*, Bd. 4, Königl. Ges. der Wiss. zu Göttingen (1880), 259-300.
- Lagrange, J.L. (1779). Sur la construction des cartes géographiques, Nouv. Mém. de l'Acad. royale des Sci. et Belle-Lett. de Berlin = *Oeuvres de Lagrange*, t. 4, Gauthier-Villars (1869), 637-692.
- Lambert, J.H. (1772). Anmerkungen und Zusätze zur Entwerfung der Land und Himmelscharten aus *Beytrage zum Gebrauche der Mathematik und deren Anwendung*, 3 Teil, Verlag der Buchhandlung der Realschule, Berlin, 105-107; expanded by A. Wangerin in *Ostwald's Klassiker der exakten Wissenschaften*, Nr 54, Leipzig (1894); English translation by W.R. Tobler: *Notes and comments on the composition of terrestrial and celestial maps*, Michigan Geographical Publications, No. 8, Ann Arbor (1972), 125 pp.
- Struik, D.J. (1961). *Lectures on classical differential geometry*, second edition, Addison-Wesley Publishing Co., Reading; reprinted by Dover Publications Inc., New York (1990).
- Vaniček, P. and Sjöberg, L. (1991). Reformulation of Stokes's theory for higher than second-degree reference field and modification of integration kernels, J. Geophysical Res. 96 B4, 6529-6539.